

Efficient quantum algorithm to construct arbitrary Dicke states

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In this paper, we study efficient algorithms towards the construction of any arbitrary Dicke state. Our contribution is to use proper symmetric Boolean functions that involve manipulations with Krawtchouk polynomials. Deutsch-Jozsa algorithm, Grover algorithm and the parity measurement technique are stitched together to devise the complete algorithm. Further, motivated by the work of Childs et al (2002), we explore how one can plug the biased Hadamard transformation in our strategy. Our work improves the results of Childs et al (2002).

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I. INTRODUCTION

Multipartite entanglement is one of the important areas in the field of quantum information that has many applications including quantum secret sharing. In this paper, we focus on the Dicke states [4], which are useful building blocks in realizing multipartite entanglement. The n -qubit weight w Dicke state, $|D_w^n\rangle$, is the equal superposition of all n -qubit states of weight w . We refer to [1, 8, 9, 11, 12, 16, 18] and the references therein for detailed discussion.

While the main focus from the viewpoint of experimental physics is to actually provide the implementation of specific Dicke states, our focus is from theoretical algorithmic angle and the only result presented in this direction appeared in [1]. In this work, we show how one can efficiently construct Dicke states by using the combinatorial properties of symmetric Boolean functions, two well-known quantum algorithms, and the generalized parity measurement. By efficient, we mean that the resource requirements in terms of quantum circuits and number of execution steps is $\text{poly}(n)$ to obtain $|D_w^n\rangle$.

Let us consider n -qubit states in the computational basis $\{0,1\}^n$ that can be written in the form $\sum_{x \in \{0,1\}^n} a_x |x\rangle$, where $\sum_{x \in \{0,1\}^n} |a_x|^2 = 1$. Thus, x can also be interpreted as a binary string and the number of 1's in the string is called the (Hamming) weight of x and denoted as $wt(x)$. Based on this an arbitrary Dicke state can be expressed as follows:

$$|D_w^n\rangle = \sum_{x \in \{0,1\}^n, wt(x)=w} \frac{1}{\sqrt{\binom{n}{w}}} |x\rangle.$$

Let us also define a symmetric n -qubit state as

$$|S^n\rangle = \sum_{x \in \{0,1\}^n} a_{wt(x)} |x\rangle, \text{ where } \sum_{i=0}^n \binom{n}{i} |a_i|^2 = 1.$$

First, we show how one can prepare a symmetric n -qubit state with the property that $\binom{n}{w} |a_w|^2$ is $\Omega(\frac{1}{\sqrt{n}})$ by using Deutsch-Jozsa algorithm [3]. This requires certain novel combinatorial observations related to symmetric Boolean functions. Then the quantum state out of Deutsch-Jozsa algorithm is measured using the parity measurement technique [8] to obtain $|D_w^n\rangle$ with a probability $\Omega(\frac{1}{\sqrt{n}})$. Thus, $O(\sqrt{n})$ runs are sufficient to obtain the required Dicke state. Note that a direct approach to construct a symmetric state has been presented in [1] using biased Hadamard transform. While the order of probability to obtain Dicke state by ours and that of [1] are the same, enumeration results show that the exact probability values are better in our case than that of [1].

Further, motivated by the idea in [1], we improve our algorithm further with a modified Deutsch-Jozsa operator that involves the biased Hadamard transform. Since biased Hadamard transform also helps to generate the target symmetric state, the overall probability to obtain the Dicke state increases.

Finally, we can also apply the Grover operator [7] before the measurement. Since Grover algorithm amplifies the amplitude of target symmetric state, this helps to reduce the necessary number of steps into $O(\sqrt[4]{n})$.

II. PROPERTIES OF SYMMETRIC BOOLEAN FUNCTIONS

A. Walsh Spectrum of Symmetric Boolean Functions

A Boolean function on n variables may be viewed as a mapping from $\{0,1\}^n$ into $\{0,1\}$. Let us denote the addition operator over $GF(2)$ by \oplus . Let $x = (x_1, \dots, x_n)$ and $\omega = (\omega_1, \dots, \omega_n)$ both belong to $\{0,1\}^n$ and the

inner product

$$x \cdot \omega = x_1 \omega_1 \oplus \cdots \oplus x_n \omega_n.$$

Let $f(x)$ be a Boolean function on n variables. Then the *Walsh transform* of $f(x)$ is a real valued function over $\{0, 1\}^n$ which is defined as

$$W_f(\omega) = \sum_{x \in \{0, 1\}^n} (-1)^{f(x) \oplus x \cdot \omega}.$$

An n -variable Boolean function f is called Symmetric if $f(x) = f(y)$ for all $x, y \in \{0, 1\}^n$ such that $wt(x) = wt(y)$. Henceforth, we will denote the set of n -variable symmetric Boolean functions as \mathcal{SB}_n .

In the truth table of $f \in \mathcal{SB}_n$, it is enough to provide outputs corresponding to different weights of elements of $\{0, 1\}^n$ only. So an n -variable symmetric function can be expressed by an $(n+1)$ length bit string as

$$ref = [f_0, f_1, \dots, f_n],$$

where f_i is the output at the inputs of weight i and ref is referred to as the simplified value vector. When $f \in \mathcal{SB}_n$, one may note that $W_f(x) = W_f(y)$ for all $x, y \in \{0, 1\}^n$ such that $wt(x) = wt(y)$. Therefore, the Walsh spectrum of f can be represented by an $(n+1)$ length integer string

$$rwf = [rwf(0), rwf(1), \dots, rwf(n)],$$

where $rwf(i)$ represents the Walsh spectrum value at the inputs of weight i .

B. Relation between Walsh Spectrum and Krawtchouk polynomials

We now relate the Walsh spectrum of the symmetric functions [17] with Krawtchouk polynomials [10, 13]. Krawtchouk polynomial of degree i is given by

$$K_i(\eta, n) = \sum_{j=0}^i (-1)^j \binom{\eta}{j} \binom{n-\eta}{i-j}.$$

From [17], we get that if $wt(\omega) = k$, then

$$W_f(\omega) = \sum_{i=0}^n (-1)^{f_i} K_i(k, n).$$

The $(n+1) \times (n+1)$ matrix which has $K_i(k, n)$ as the (i, k) -th element is known as the Krawtchouk matrix [5, 6].

For example, let us present the Krawtchouk matrix for $n = 5$ and 6 as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 3 & 1 & -1 & -3 & -5 \\ 10 & 2 & -2 & -2 & 2 & 10 \\ 10 & -2 & -2 & 2 & 2 & -10 \\ 5 & -3 & 1 & 1 & -3 & 5 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 0 & -2 & -4 & -6 \\ 15 & 5 & -1 & -3 & -1 & 5 & 15 \\ 20 & 0 & -4 & 0 & 4 & 0 & -20 \\ 15 & -5 & -1 & 3 & -1 & -5 & 15 \\ 6 & -4 & 2 & 0 & -2 & 4 & -6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

In these two matrices, one can verify the properties related to the Krawtchouk matrix given in Proposition 1.

To determine all the Walsh spectrum values of $f \in \mathcal{SB}_n$, it is enough to multiply $((-1)^{f_0}, \dots, (-1)^{f_n})$ with the $(n+1) \times (n+1)$ Krawtchouk matrix. Applying Krawtchouk matrix, the analysis of the Walsh spectra of symmetric functions becomes combinatorially interesting. Elements of a Krawtchouk matrix have nice combinatorial properties and they follow nice symmetry [10] too. We list some of them in the following proposition.

- Proposition 1**
1. $K_0(k, n) = 1, K_1(k, n) = n - 2k,$
 2. $(i+1)K_{i+1}(k, n) = (n - 2k)K_i(k, n) - (n - i + 1)K_{i-1}(k, n),$
 3. $K_i(k, n) = (-1)^k K_{n-i}(k, n),$
 4. $\binom{n}{k} K_i(k, n) = \binom{n}{i} K_k(i, n),$
 5. $K_i(k, n) = (-1)^i K_i(n - k, n),$
 6. $(n - k)K_i(k + 1, n) = (n - 2i)K_i(k, n) - kK_i(k - 1, n),$
 7. $(n - i + 1)K_i(k, n + 1) = (3n - 2i - 2k + 1)K_i(k, n) - 2(n - k)K_i(k, n - 1).$

C. Implementation of Symmetric Boolean Functions

The symmetric Boolean functions can be efficiently implemented. As described in [2], the circuit complexity of n -variable symmetric Boolean functions is $4.5n + o(n)$. It is known that given a classical circuit f , there is a quantum circuit of comparable efficiency which computes the transformation U_f that takes input like $|x, y\rangle$ and produces output like $|x, y \oplus f(x)\rangle$. Thus, we will consider that for $f \in \mathcal{SB}_n$, the quantum circuit U_f can be efficiently implemented using $O(n)$ circuit complexity.

III. ALGORITHMS

A. Find a Special Symmetric Boolean Function which maximizes the Walsh Spectrum

Consider that we want to maximize the Walsh spectrum value corresponding to weight w points and naturally, from the property of symmetric functions, all of them will be equal. Now we present an important combinatorial result to show how to find such symmetric Boolean functions.

Theorem 1 Consider $f \in \mathcal{SB}_n$. The function f , represented as ref , for which the Walsh spectrum correspond-

ing to the w weight points will be maximized, can be written as

$$f_i = \begin{cases} 0, & \text{if } K_i(w, n) > 0 \\ 1, & \text{if } K_i(w, n) < 0 \\ 0 \text{ or } 1, & \text{if } K_i(w, n) = 0 \end{cases}$$

Proof: We have $W_f(\omega) = \sum_{i=0}^n (-1)^{f_i} K_i(k, n)$. One may note that the maximum value of $\sum_{i=0}^n (-1)^{f_i} K_i(k, n)$ is $\sum_{i=0}^n |K_i(k, n)|$. This is attained when we take the function of the form as described in the theorem. ■

Example 1 As example, consider $n = 6, w = k = 2$. In the corresponding column of the $(6+1) \times (6+1)$ matrix, we get the values as 1, 2, -1, -4, -1, 2, 1. Thus, we will consider the function with ref as $[0, 0, 1, 1, 1, 0, 0]$. For such an $f \in \mathcal{SB}_6$, the Walsh spectrum values at the points ω , such that $w = wt(\omega) = 2$, will be maximized, which is $1 + 2 + 1 + 4 + 1 + 2 + 1 = 12$.

B. Walsh Spectrum of the Special Symmetric Boolean function by Combinatorial Property of Krawtchouk Matrix

Next we present certain results related to column sum of Krawtchouk matrix.

Lemma 1 $\sum_{i=0}^n |K_i(\lceil \frac{n}{2} \rceil, n)| = \sum_{i=0}^n |K_i(\lfloor \frac{n}{2} \rfloor, n)| = 2^{\lceil \frac{n}{2} \rceil}$.

Proof: Let us first prove this for even n .

Following Proposition 1(2), we have

$$(i+1)K_{i+1}(k, n) = (n-2k)K_i(k, n) - (n-i+1)K_{i-1,n}(k, n).$$

For n even, and $k = \frac{n}{2}$, we get,

$$K_{i+1}(\frac{n}{2}, n) = -\frac{n-i+1}{i+1} K_{i-1,n}(\frac{n}{2}, n).$$

That is, the recurrence relation follows:

$$K_i(\frac{n}{2}, n) = -\frac{n-i+2}{i} K_{i-2,n}(\frac{n}{2}, n),$$

with the initial conditions $K_0(\frac{n}{2}, n) = 1$ and $K_1(\frac{n}{2}, n) = 0$ as available from Proposition 1(1). Thus one may note that for odd i , $K_i(\frac{n}{2}, n) = 0$. Further, using induction, for even i , we get

$$|K_i(\frac{n}{2}, n)| = \left(\frac{n}{2}\right)_i.$$

Thus,

$$\sum_{i=0}^n |K_i(\frac{n}{2}, n)| = \sum_{l=0}^{\frac{n}{2}} \left(\frac{n}{2}\right)_l,$$

putting $i = 2l$. Hence,

$$\sum_{i=0}^n |K_i(\frac{n}{2}, n)| = 2^{\frac{n}{2}}.$$

Now let us prove this for odd n .

For n odd and $k = \frac{n-1}{2}$, from Proposition 1(2) we get

$$K_{i+1}(\frac{n-1}{2}, n) = \frac{1}{i+1} K_i(\frac{n-1}{2}, n) - \frac{n-i+1}{i+1} K_{i-1}(\frac{n-1}{2}, n).$$

That is, the recurrence relation is as follows:

$$K_i(\frac{n-1}{2}, n) = \frac{1}{i} K_{i-1}(\frac{n-1}{2}, n) - \frac{n-i+2}{i} K_{i-2}(\frac{n-1}{2}, n).$$

One can now show by induction that

$$K_{2i}(\frac{n-1}{2}, n) = K_{2i+1}(\frac{n-1}{2}, n), \quad \forall i, 1 \leq i \leq \frac{n-1}{2}.$$

Using the above two identities and induction, one can verify that $|K_{2l}(\frac{n-1}{2}, n)| = \left(\frac{n-1}{2}\right)_l$. Thus,

$$\sum_{i=0}^n |K_i(\lfloor \frac{n}{2} \rfloor, n)| = 2 \sum_{l=0}^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)_l,$$

where $i = 2l$. Hence, we get,

$$\sum_{i=0}^n |K_i(\lfloor \frac{n}{2} \rfloor, n)| = 2 \cdot 2^{\frac{n-1}{2}} = 2^{\frac{n+1}{2}}.$$

Using Proposition 1(5), we get that

$$\sum_{i=0}^n |K_i(\lfloor \frac{n}{2} \rfloor, n)| = \sum_{i=0}^n |K_i(\lceil \frac{n}{2} \rceil, n)|.$$

That completes the proof. ■

Theorem 2 Let $f \in \mathcal{SB}_n$ be as explained in Theorem 1 towards maximizing the Walsh spectrum values at weight $\lceil \frac{n}{2} \rceil$ or $\lfloor \frac{n}{2} \rfloor$. Then,

$$\left(\frac{n}{\lceil \frac{n}{2} \rceil}\right) (rw_f(\lceil \frac{n}{2} \rceil))^2 = \left(\frac{n}{\lfloor \frac{n}{2} \rfloor}\right) (rw_f(\lfloor \frac{n}{2} \rfloor))^2 \text{ is } \Omega\left(\frac{2^{2n}}{\sqrt{n}}\right).$$

Proof: The Walsh spectrum in this case is

$$\begin{aligned} rw_f(\lceil \frac{n}{2} \rceil) &= \sum_{i=0}^n |K_i(\lceil \frac{n}{2} \rceil, n)| \\ &= rw_f(\lfloor \frac{n}{2} \rfloor) = \sum_{i=0}^n |K_i(\lfloor \frac{n}{2} \rfloor, n)|. \end{aligned}$$

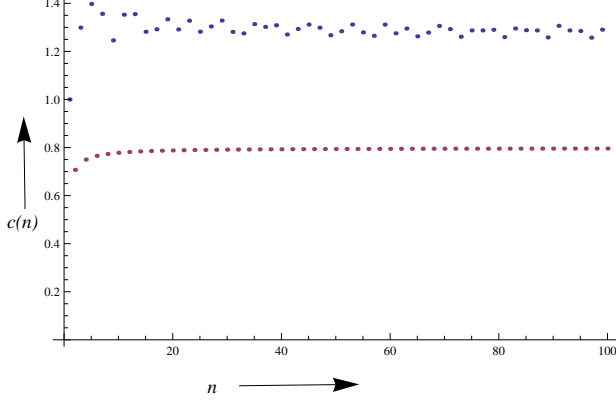


FIG. 1: Plot of $c(n)$ vs n for $1 \leq n \leq 100$.

Thus the total sum of the squares of the Walsh spectrum values at weight $\lceil \frac{n}{2} \rceil$ or $\lfloor \frac{n}{2} \rfloor$ is

$$\binom{n}{\lceil \frac{n}{2} \rceil} \left(\sum_{i=0}^n |K_i(\lceil \frac{n}{2} \rceil, n)| \right)^2$$

which is $\Omega(\frac{2^{2n}}{\sqrt{n}})$, by Stirling's approximation. ■

One may similarly note that for the trivial cases of $w = 0$ or n , if one chooses $f \in \mathcal{SB}_n$ following Theorem 1, then $\binom{n}{w} (rw_f(w))^2 = 2^{2n}$. However, proving the result similar to Theorem 2 for any n and any weight w , in general, seems to be quite tedious. Thus we make detailed enumerations to obtain $c(n) = \min_{w=0}^n \frac{\binom{n}{w} (rw_f(w))^2}{2^{2n}}$ that has been verified for $n \leq 1000$ and we note that the values stabilize as $c(999) = 1.24793$ and $c(1000) = 0.797685$. The graph of this is plotted in Figure 1 for $1 \leq n \leq 100$, the points for odd n are coming above and those for even n are coming below. Since we are not providing a proof of this, we refer this as follows.

Fact 1 Let $f \in \mathcal{SB}_n$ be as explained in Theorem 1 towards maximizing the Walsh spectrum values at weight w . Then the total sum of the squares of the Walsh spectrum values at weight w , $\binom{n}{w} (rw_f(w))^2$, is $\Omega(\frac{2^{2n}}{\sqrt{n}})$.

C. Relation between Deutsch-Jozsa algorithm and the Walsh Spectrum of Symmetric Boolean Function

Given f is either constant or balanced, if the corresponding quantum implementation U_f is available, Deutsch-Jozsa [3] provided a quantum algorithm that decide in constant time which one it is. Let us now describe our interpretation of Deutsch-Jozsa algorithm in terms of Walsh spectrum values. We denote the operator for Deutsch-Jozsa algorithm as

$$\mathcal{D}_f = H^{\otimes n} U_f H^{\otimes n},$$

where the Boolean function f is available as an oracle U_f . For brevity, we abuse the notation and do not write the auxiliary qubit, i.e., $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ and the corresponding output in this case.

Now one can observe that [14]

$$\begin{aligned} \mathcal{D}_f |0\rangle^{\otimes n} &= \sum_{z \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} \frac{(-1)^{x \cdot z \oplus f(x)}}{2^n} |z\rangle \\ &= \sum_{z \in \{0,1\}^n} \frac{W_f(z)}{2^n} |z\rangle. \end{aligned}$$

Note that the associated probability with a state $|z\rangle$ is $\frac{W_f^2(z)}{2^{2n}}$. Hence we have the following technical result as pointed out in [14] with our interpretation for symmetric functions.

Proposition 2 Given an n -variable Boolean function f , $\mathcal{D}_f |0\rangle^{\otimes n}$ produces a superposition of all states $z \in \{0,1\}^n$ with the amplitude $\frac{W_f(z)}{2^n}$ corresponding to each state $|z\rangle$. Specially, if $f \in \mathcal{SB}_n$, then the amplitude corresponding to $|z\rangle$ is $\frac{\sum_{i=0}^n (-1)^{f_i} K_i(wt(z), n)}{2^n}$.

D. Algorithm 1: Deutsch-Jozsa Algorithm with Special Symmetric Function

Based on the overall properties, we propose a quantum algorithm as shown in the following algorithm.

Algorithm 1

1. Choose $f \in \mathcal{SB}_n$ as explained in Theorem 1 to maximize the Walsh spectrum values at weight w .
2. Use the Deutsch-Jozsa algorithm to obtain a symmetric n -qubit state $|S^n\rangle = \sum_{x \in \{0,1\}^n} a_{wt(x)} |x\rangle$, such that $\binom{n}{w} |a_w|^2$ is $\Omega(\frac{1}{\sqrt{n}})$.
3. Apply the parity measurement strategy. If the ancilla state is measured at the basis $U^w |\zeta\rangle$, then $|D_w^n\rangle$ is successfully obtained. Else go to Step 2 and iterate.

The following result provides the estimate of complexity of our algorithm.

Theorem 3 Let $f \in \mathcal{SB}_n$ be as explained in Theorem 1 towards maximizing the Walsh spectrum values at weight w . Given Fact 1, Deutsch-Jozsa algorithm produces a symmetric n -qubit state (before the measurement) $|S^n\rangle = \sum_{x \in \{0,1\}^n} a_{wt(x)} |x\rangle$, such that $\binom{n}{w} |a_w|^2$ is $\Omega(\frac{1}{\sqrt{n}})$.

Proof: The proof follows from Theorem 1, Fact 1 and Proposition 2. ■

Now the final step is to measure the symmetric state until we get the target Dicke state by using parity measurement method [8, Section IIIA]. Now we explain how to exploit the parity measurement in our purpose. Note

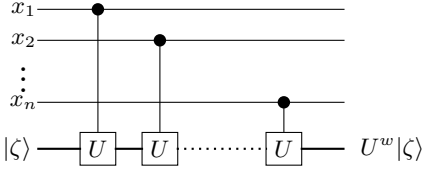


FIG. 2: Generalized Parity Module as in [8, Fig. 1]. If $wt(x) = w$, then the ancilla will become $U^w|\zeta\rangle$.

[8, Section IIIA] assumes to use n dimensional qudit ancilla, but we consider a qudit $|\zeta\rangle$ of dimension $n+1$ here. A certain unitary operator U is designed such that,

$$|\zeta\rangle, U|\zeta\rangle, U^2|\zeta\rangle, \dots, U^{n-1}|\zeta\rangle, U^n|\zeta\rangle$$

are all orthogonal to each other and $U^{n+1}|\zeta\rangle = |\zeta\rangle$. Since $|\zeta\rangle$ is an $(n+1)$ -dimensional state, one can indeed obtain a set of such $n+1$ orthogonal states. The parity measurement is done on the

$$\{|\zeta\rangle, U|\zeta\rangle, U^2|\zeta\rangle, \dots, U^n|\zeta\rangle\}$$

basis. Here $|\zeta\rangle$ is used as the target state. For the n -qubit control state $|x\rangle$, if it has weight w then its corresponding target state, after application of this circuit, will become $U^w|\zeta\rangle$ (see Figure 2). Now consider an n -qubit symmetric state as the control input, which is $|\tau\rangle = \sum_{x \in \{0,1\}^n} a_w |x\rangle$,

where $w = wt(x)$, $\sum_{i=0}^n \binom{n}{i} |a_i|^2 = 1$. After applying this circuit, one obtains $\sum_{x \in \{0,1\}^n} a_w |x\rangle U^w|\zeta\rangle$. Thus, if one measures in

$$\{|\zeta\rangle, U|\zeta\rangle, U^2|\zeta\rangle, \dots, U^n|\zeta\rangle\}$$

basis, then the state $|D_w^n\rangle$ will be obtained when the measurement result of the ancilla state is $U^w|\zeta\rangle$.

Since the probability of target Dicke state is $\Omega(\frac{1}{\sqrt{n}})$, we should repeat the whole procedure at most $O(\sqrt{n})$.

Example 2 Let us have an example taking $n = 6, w = 2$ to outline our method. In this case the Dicke state will be

$$|D_2^6\rangle = \sum_{x \in \{0,1\}^6, wt(x)=2} \frac{1}{\sqrt{15}} |x\rangle.$$

We start with is an $n = 6$ variable symmetric Boolean function having Walsh spectrum value at each of the weight $w = 2$ point as 12 (following Theorem 1, one may refer to Example 1 also). There are $\binom{n}{w} = \binom{6}{2} = 15$ such points. The amplitude associated with each point after the DJ algorithm is $\frac{12}{2^6} = \frac{3}{16}$. Thus we get $\sum_{x \in \{0,1\}^6, wt(x)=2} \frac{3}{16} |x\rangle + \sum_{x \in \{0,1\}^6, wt(x) \neq 2} a_x |x\rangle$ initially. Thus, the probability associated with $|D_2^6\rangle$ will be $\binom{6}{2} (\frac{3}{16})^2 = \frac{135}{256} = 0.52734375$ and hence one may note that the probability that, after parity measurement, it will land into $|D_2^6\rangle$ is quite high.

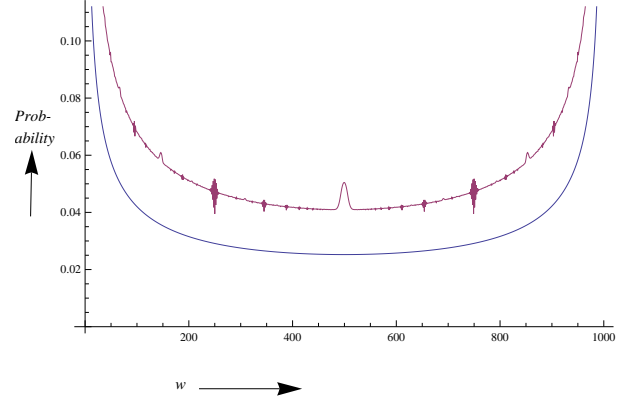


FIG. 3: Plot of probabilities associated with $|D_w^n\rangle$ against w in our case (above) and in [1] (below) for $n = 999$.

E. Comparison with a Previous Method

It was explained in [8] how one can obtain $|D_w^n\rangle$ from $\frac{1}{2^{\frac{n}{2}}} \sum_{x \in \{0,1\}^n} |x\rangle$. However, the idea explained in [8, Section IIIA] works efficiently only for $w = \lceil \frac{n}{2} \rceil$ or $\lfloor \frac{n}{2} \rfloor$. The most general work in this direction has appeared in [1] where biased Hadamard transformation was exploited. The strategy of [1] uses biased Hadamard transformation

$$\left(\begin{array}{cc} \sqrt{1 - \frac{w}{n}} & \sqrt{\frac{w}{n}} \\ \sqrt{\frac{w}{n}} & -\sqrt{1 - \frac{w}{n}} \end{array} \right)^{\otimes n}$$

on $|0\rangle^{\otimes n}$ such that the probability associated with $|D_w^n\rangle$ will be $\binom{n}{w} (\frac{w}{n})^w (1 - \frac{w}{n})^{n-w} \geq \sqrt{\frac{2}{n\pi}}$, i.e., $\Omega(\frac{1}{\sqrt{n}})$. Thus, the probability of our case and also in [1] are of the same order. While the theoretical comparison of the exact probability values seems elusive, we have made detailed enumerations to observe that the exact probability values in our case are better than that of [1]. Below we present enumeration results towards that.

First we present two graphs to show the probability values associated to $|D_w^n\rangle$ for all w , when $n = 999$ (to represent odd case) or 1000 (to represent even case). For our case, it is $\binom{n}{w} \frac{(\sum_{i=0}^n |K_i(w,n)|)^2}{2^{2n}}$ (after application of Deutsch-Jozsa algorithm without measurement), and for the case of [1] it is $\binom{n}{w} (\frac{w}{n})^w (1 - \frac{w}{n})^{n-w}$ (after application of biased Hadamard transform). From Figure 3 and Figure 4, it is clear to note that our method provides higher probability (the upper curve) in all the cases except $w = 0, n$ (which are trivial ones) and $w = \frac{n}{2}$ for $n = 1000$.

In both figures, the present algorithm shows some variation of probability when the weight is around $\lfloor \frac{n}{4} \rfloor$. To check whether or not these cases still shows the higher probability than the previous method, we look into the $w = \lfloor \frac{n}{4} \rfloor$ case a little bit more. From Figure 5, one may

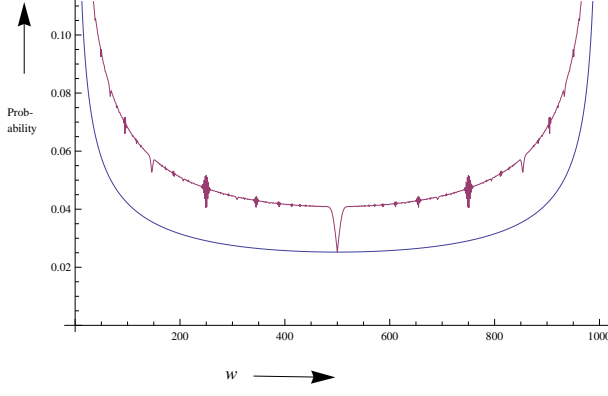


FIG. 4: Plot of probabilities associated with $|D_w^n\rangle$ against w in our case (above) and in [1] (below) for $n = 1000$.

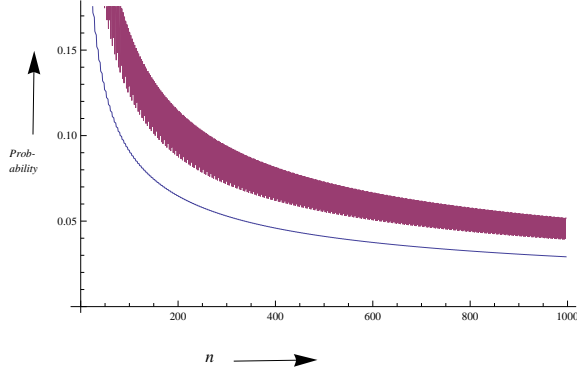


FIG. 5: Plot of probabilities associated with $|D_w^n\rangle$ against n in our case (above) and in [1] (below) for $n = 4$ to 1000 and $w = \lfloor \frac{n}{4} \rfloor$.

note that our probability values (the upper curve) are higher than the case of [1]. These results explain the advantage of the use of a suitable symmetric Boolean function which shows higher Walsh spectrum values for the given weight.

F. Algorithm 2: Additional Improvement by exploiting biased Hadamard Operator

We have provided numerical evidences that using proper symmetric Boolean functions in Deutsch-Jozsa algorithm provides better probability than the use of biased Hadamard transform as described in [1]. However, motivated by [1], a natural extension should be to couple biased Hadamard transform in Deutsch-Jozsa algorithm

instead of (unbiased) Hadamard transform. Thus, let us refer to the general description of a Hadamard type transformation (biased or unbiased) that can be written as $B_{r,n} = \begin{pmatrix} \sqrt{1-\frac{r}{n}} & \sqrt{\frac{r}{n}} \\ \sqrt{\frac{r}{n}} & -\sqrt{1-\frac{r}{n}} \end{pmatrix}$. We will replace the standard notation of w here by r as we will not restrict ourselves to integer values $w \in [0, \dots, n]$, but use any real number $r \in [0, n]$ to obtain the optimum probability of success to get a Dicke state.

Instead of using the operator $\mathcal{D}_f = H^{\otimes n} U_f H^{\otimes n}$, let us first describe the most general operator of the form

$$\mathcal{D}'_f = B_{r_1,n}^{\otimes n} U_f B_{r_2,n}^{\otimes n}, \quad (1)$$

where r_1, r_2 are real numbers in $[0, n]$.

First, we consider the case when $r_1 = \frac{n}{2}$, i.e., $B_{r_1,n} = H$, but $r_2 = r$ varies towards optimization. That is

$$\mathcal{D}'_f = H^{\otimes n} U_f B_{r,n}^{\otimes n}. \quad (2)$$

One may note that the application of \mathcal{D}'_f on $|0\rangle^{\otimes n}$ will produce

$$\mathcal{D}'_f |0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} \left(\sum_{x \in \{0,1\}^n} (-1)^{x \cdot z \oplus f(x)} \left(1 - \frac{r}{n}\right)^{\frac{n-d(x,z)}{2}} \left(\frac{r}{n}\right)^{\frac{d(x,z)}{2}} \right) |z\rangle,$$

where $d(x, z)$ is the (Hamming) distance between two same length binary strings x and z . Before proceeding further, we also have the following technical result.

Proposition 3 Let $\mathcal{D}'_f = H^{\otimes n} U_f B_{r,n}^{\otimes n}$. If $f \in \mathcal{SB}_n$ then $\mathcal{D}'_f |0\rangle^{\otimes n}$ is a symmetric state.

Proof: We need to prove that $\sum_{x \in \{0,1\}^n} (-1)^{x \cdot z \oplus f(x)} \left(1 - \frac{r}{n}\right)^{\frac{n-d(x,z)}{2}} \left(\frac{r}{n}\right)^{\frac{d(x,z)}{2}}$ is same for all the $z \in \{0,1\}^n$ having the same Hamming weight. Let us consider $u, v \in \{0,1\}^n$ such that $u \neq v$, but $wt(u) = wt(v)$. Then we need to prove that $\sum_{x \in \{0,1\}^n} (-1)^{x \cdot u \oplus f(x)} \left(1 - \frac{r}{n}\right)^{\frac{n-d(x,u)}{2}} \left(\frac{r}{n}\right)^{\frac{d(x,u)}{2}} = \sum_{x \in \{0,1\}^n} (-1)^{x \cdot v \oplus f(x)} \left(1 - \frac{r}{n}\right)^{\frac{n-d(x,v)}{2}} \left(\frac{r}{n}\right)^{\frac{d(x,v)}{2}}$.

The proof follows from the fact that $\sum_{x \in \{0,1\}^n, wt(x)=w} (-1)^{x \cdot u \oplus f(x)} \left(1 - \frac{r}{n}\right)^{\frac{n-d(x,u)}{2}} \left(\frac{r}{n}\right)^{\frac{d(x,u)}{2}} = \sum_{x \in \{0,1\}^n, wt(x)=w} (-1)^{x \cdot v \oplus f(x)} \left(1 - \frac{r}{n}\right)^{\frac{n-d(x,v)}{2}} \left(\frac{r}{n}\right)^{\frac{d(x,v)}{2}}$, given that f is symmetric. ■

The main problem in this case is that we need to go for trial and error by modifying the symmetric Boolean functions and trying out different values of $\frac{r}{n}$. So far, we could not obtain the exact characterization of symmetric functions while biased Hadamard transform is used. Based on this, we propose Algorithm 2 as follows.

Algorithm 2

1. Apply \mathcal{D}'_f to $|0\rangle^{\otimes n}$ to obtain a symmetric n -qubit state. The value of $\frac{r}{n}$ and the choice of the symmetric Boolean functions are achieved heuristically.
2. Use parity measurement strategy. If the ancilla state is measured at the basis $U^w |\zeta\rangle$, then $|D_w^n\rangle$ is successfully obtained. Else take the parameters as in Step 1 once again and iterate Step 2.

As we could not characterize this, to provide some experimental results in this direction (see Table I), we used the following method for some small values of n ($n = 4$ to 9). We select each of the Boolean functions f from \mathcal{SB}_n . Given f and a specific weight w , $1 \leq w \leq n$, we write the expression of success probability as a function of r . Then we apply the *Maximize* function available in *Mathematica* 8.0 to compute the optimum value of r given f, w , so that the success probability becomes maximum. Note that, as we could not characterize the process yet, this is an exhaustive task and for each n , it requires checking of 2^{n+1} symmetric Boolean functions. That is the reason, we can study it for only a few small values of n . However, this is a classical computation that can be done as an off-line work. Once such programs are executed, we can have a database of proper $f \in \mathcal{SB}_n$ and the corresponding r to have the optimal success probability to obtain $|D_w^n\rangle$. Given these data, the actual quantum algorithm to obtain Dicke states can be efficiently implemented.

IV. NUMERICAL COMPARISON OF THREE APPROACHES

Now we compare three approaches:

- using biased Hadamard operator as shown in [1],
- Algorithm 1 based on $\mathcal{D}_f|0\rangle^{\otimes n}$, and
- Algorithm 2 based on $\mathcal{D}'_f|0\rangle^{\otimes n}$.

The first two cases need $O(\sqrt{n})$ complexity, and the third one is a heuristic that shows improved results than the first two. Some numerical results of the probability associated with $|D_w^n\rangle$ are shown in Table I for $n = 4, \dots, 9$. As shown in the table, we note that Algorithm 2 provides the highest probability than others. There are a few interesting observations from the enumeration results.

- We note that the improvements using \mathcal{D}'_f are highly significant at $w = \lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$ and the significance reduces as w moves away from the middle, i.e., towards $w = 1$ or $n - 1$.
- In case of using \mathcal{D}'_f , the success probabilities at w and $n - w$ weights are same for all the values of w , i.e., $w \geq 1$. However, the values of r in those cases are same at w and $n - w$ weights for $w \geq 2$ only.

V. ADDITIONAL IMPROVEMENT BY APPLYING GROVER ALGORITHM

A. Additional Use of Grover Algorithm

Although we can construct the target Dicke state by measuring the intermediate quantum state, we may in-

crease the efficiency further by using the amplitude amplification method. Based on this, an adiabatic evolution has been used towards amplitude amplification of the desired states in [1], but no complexity analysis was shown. In this work, instead, we apply the conventional Grover algorithm [7] as it provides a quadratic speed-up.

Instead of equal superposition $|\psi\rangle = H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle$ in Grover algorithm, we will use the symmetric state of the form $|\Psi\rangle = \mathcal{D}_f(|0\rangle^{\otimes n}) = \sum_{x \in \{0,1\}^n} \frac{W_f(x)}{2^n} |x\rangle$ exploiting the properly chosen Boolean function $f(x)$, as explained in the previous sections.

Further, towards inverting the phase, we will use another symmetric Boolean function $g(x)$, different from $f(x)$, where $g(x) = 1$, when $wt(x) = w$, and $g(x) = 0$, otherwise. Based on $g(x)$, we implement the inversion operator as \mathcal{O}_g , that inverts the phase of the states $|x\rangle$ where $\{x \in \{0,1\}^n | wt(x) = w\}$. Thus, we consider the operator

$$G_t = [(2|\Psi\rangle\langle\Psi| - I)\mathcal{O}_g]^t$$

on $|\Psi\rangle$ to get $|\Psi_t\rangle$.

Consider the n -qubit state $|\Psi\rangle = \sum_{s \in S} u_s |s\rangle + \sum_{s \in \{0,1\}^n \setminus S} v_s |s\rangle$, where u_s, v_s are real and $\sum_{s \in S} u_s^2 + \sum_{s \in \{0,1\}^n \setminus S} v_s^2 = 1$. For brevity, let us represent $|\Psi\rangle = \sum_{s \in S} u_s |s\rangle + \sum_{s \in \{0,1\}^n \setminus S} v_s |s\rangle = u|X\rangle + v|Y\rangle$. That is, $u^2 = \sum_{s \in S} u_s^2$ and $v^2 = \sum_{s \in \{0,1\}^n \setminus S} v_s^2$.

Theorem 4 *Let $u = \sin\theta$, $v = \cos\theta$. The application of $[(2|\Psi\rangle\langle\Psi| - I)\mathcal{O}_g]^t$ operator on $|\Psi\rangle$ produces $|\Psi_t\rangle$, in which the probability amplitude of $|X\rangle$ is $\sin(2t + 1)\theta$.*

Proof: For $t = 1$, one can check that $|\Psi_1\rangle = [(2|\Psi\rangle\langle\Psi| - I)\mathcal{O}_g]|\Psi\rangle = [(2|\Psi\rangle\langle\Psi|)\mathcal{O}_g]|\Psi\rangle - \mathcal{O}_g|\Psi\rangle$. Now substituting the values of u, v , we get that $|\Psi_1\rangle = \sin 3\theta|X\rangle + \cos 3\theta|Y\rangle$.

Now we will use induction. Let the application of $[(2|\Psi\rangle\langle\Psi| - I)\mathcal{O}_g]^t$ operator on $|\Psi\rangle$ updates the probability amplitude of $|X\rangle$ as $\sin(2t\theta + \theta)$, for $t = k$. From the assumption we have $[(2|\Psi\rangle\langle\Psi| - I)\mathcal{O}_g]^k|\Psi\rangle = \sin(\theta + 2k\theta)|X\rangle + \cos(\theta + 2k\theta)|Y\rangle$. Now, for $t = k + 1$, it can be checked that $[(2|\Psi\rangle\langle\Psi| - I)\mathcal{O}_g]^{(k+1)}|\Psi\rangle = \sin(\theta + 2(k + 1)\theta)|X\rangle + \cos(\theta + 2(k + 1)\theta)|Y\rangle$. Thus, the proof. ■

We will now use such states $|\Psi_t\rangle$ in parity measurement. Consider that after the Deutsch-Jozsa algorithm we obtain a symmetric n -qubit state (before the measurement)

$$|S^n\rangle = \sum_{x \in \{0,1\}^n} a_{wt(x)} |x\rangle,$$

such that $\binom{n}{w}|a_w|^2 = \frac{c}{\sqrt{n}}$, for some constant c . Thus, we have the initial amplitude of target states, $\{x \in \{0,1\}^n | wt(x) = w\}$, is $\sin\theta = \binom{n}{w}|a_w| = \sqrt{\frac{c}{\sqrt{n}}}$. For

$n \downarrow$	$w \rightarrow$	1	2	3	4	5	6	7	8
4	$\mathcal{D}'_f 0\rangle^{\otimes n}$	0.833609	0.981763	0.833609	—	—	—	—	—
	f	01	02	05	—	—	—	—	—
	r	0.468136	0.298698	0.468136	—	—	—	—	—
	$\mathcal{D}_f 0\rangle^{\otimes n}$	0.5625	0.375	0.5625	—	—	—	—	—
	[1]	0.421875	0.375	0.421875	—	—	—	—	—
5	$\mathcal{D}'_f 0\rangle^{\otimes n}$	0.748304	0.92852	0.92852	0.748304	—	—	—	—
	f	03	02	05	16	—	—	—	—
	r	1.42458	0.313077	0.313077	3.57542	—	—	—	—
	$\mathcal{D}_f 0\rangle^{\otimes n}$	0.703125	0.625	0.625	0.703125	—	—	—	—
	[1]	0.4096	0.3456	0.3456	0.4096	—	—	—	—
6	$\mathcal{D}'_f 0\rangle^{\otimes n}$	0.730278	0.823495	0.954987	0.823495	0.730278	—	—	—
	f	03	02	05	0A	29	—	—	—
	r	1.48129	0.357282	0.277975	0.357282	4.51871	—	—	—
	$\mathcal{D}_f 0\rangle^{\otimes n}$	0.585938	0.527344	0.3125	0.527344	0.585938	—	—	—
	[1]	0.401878	0.329218	0.3125	0.329218	0.401878	—	—	—
7	$\mathcal{D}'_f 0\rangle^{\otimes n}$	0.704306	0.754753	0.907588	0.907588	0.754753	0.704306	—	—
	f	07	60	05	0A	53	4A	—	—
	r	2.44507	5.93733	0.27984	0.27984	5.93733	2.44507	—	—
	$\mathcal{D}_f 0\rangle^{\otimes n}$	0.683594	0.512695	0.546875	0.546875	0.512695	0.683594	—	—
	[1]	0.396569	0.318745	0.293755	0.293755	0.318745	0.396569	—	—
8	$\mathcal{D}'_f 0\rangle^{\otimes n}$	0.698181	0.710643	0.813922	0.92625	0.813922	0.710643	0.698181	—
	f	3F	C0	BF	A0	AF	AC	AD	—
	r	5.51859	6.91248	7.69903	7.74472	7.69903	6.91248	5.51859	—
	$\mathcal{D}_f 0\rangle^{\otimes n}$	0.598145	0.553711	0.413574	0.273438	0.413574	0.553711	0.598145	—
	[1]	0.392696	0.311462	0.281632	0.273438	0.281632	0.311462	0.392696	—
9	$\mathcal{D}'_f 0\rangle^{\otimes n}$	0.684842	0.651002	0.76886	0.884277	0.884277	0.76886	0.651002	0.684842
	f	0F	180	0D	140	15F	6A	153	16A
	r	3.4566	7.86171	0.858163	8.7469	8.7469	0.858153	7.86171	3.4566
	$\mathcal{D}_f 0\rangle^{\otimes n}$	0.672913	0.430664	0.415283	0.492188	0.492188	0.415283	0.430664	0.672913
	[1]	0.389744	0.306102	0.273129	0.260182	0.260182	0.273129	0.306102	0.389744

TABLE I: Probability values using biased Hadamard transform (in this case we provide the corresponding symmetric function f represented as a hexadecimal number of the $(n+1)$ length bit-string $f_n, f_{n-1}, \dots, f_1, f_0$ and the value of r), using (standard) Hadamard transform and the method of [1].

large n , one can approximate it as $\theta = \frac{\sqrt{c}}{\sqrt[4]{n}}$ and hence we need t iterations of Grover like strategy such that $(2t+1)\theta \approx \frac{\pi}{2}$, i.e., $t \approx \frac{\pi \sqrt[4]{n}}{2\sqrt{c}}$.

One important difference with Grover algorithm here is that we have good (almost exact) estimate of t , which is not known priori for application in search algorithms. After the application of Grover like strategy, we will get another symmetric n -qubit state $|T^n\rangle = \sum_{x \in \{0,1\}^n} a'_{wt(x)} |x\rangle$ such that $\binom{n}{w} |a'_w|^2$ will be very

close to 1 and the parity measurement will produce $|D_w^n\rangle$ mostly in one step with very high probability. Thus the exact strategy is similar to Algorithm 1 (Algorithm 2 can be modified with a similar way) in the previous section, where we add one more step as follows.

Algorithm 3

1. Let $f \in \mathcal{SB}_n$ be as explained in Theorem 1 to maximize the Walsh spectrum values at weight w .
2. Use any of the above three strategies (our strategies exploiting Hadamard or biased Hadamard transform or the strategy of [1]) to obtain a symmetric n -qubit state $|S^n\rangle = \sum_{x \in \{0,1\}^n} a_{wt(x)} |x\rangle$, such that $\binom{n}{w} |a_w|^2$ is $\Omega(\frac{1}{\sqrt[4]{n}})$.
3. Use G_t on $|S^n\rangle$, t many times, where t is $O(\sqrt[4]{n})$ to obtain $|T^n\rangle = \sum_{x \in \{0,1\}^n} a'_{wt(x)} |x\rangle$ such that $\binom{n}{w} |a'_w|^2$ is very close to 1.
4. Use parity measurement strategy. If the ancilla state is measured at the basis $U^w|\zeta\rangle$, then $|D_w^n\rangle$ is successfully obtained. Else go to Step 2 and iterate.

Example 3 Let us consider $n = 8, w = \frac{n}{2} = 4$. One may note that using our technique in Section III D or using the technique of [1], one can obtain $|D_w^n\rangle$ with probability $\frac{\binom{n}{\frac{n}{2}}}{2^n} = 0.2734375$. Thus, $\sin \theta = \sqrt{\frac{\binom{n}{\frac{n}{2}}}{2^n}} \approx 0.523$. Taking, $t = 1$, one may note that $\sin(2t + 1)\theta = \sin 3\theta \approx 0.9968$. Hence after the application of Grover algorithm with $t = 1$, the probability of obtaining $|D_w^n\rangle$ increases to 0.9968.

B. Potential Advantage

The advantage of this method over Algorithm 1 is as follows. First, we need $O(\sqrt[4]{n})$ steps using Grover algorithm in each run and then a parity measurement should provide $|D_w^n\rangle$. Thus, we get a quadratic speed-up (which is quite natural) over just using Deutsch-Jozsa algorithm. Second, the number of parity measurement was $O(\sqrt{n})$ in the earlier case, once in each iteration. Here it is only a very few (may be 1 in most of the cases).

It is natural to use Grover algorithm for amplitude amplification, but in the known applications, the number of target states for which the amplitude is increased are not large. That guarantees the efficient implementation of the phase reversal circuit. In the case of preparing $|D_w^n\rangle$, the situation is different as we need to amplify the amplitude at $\binom{n}{w}$ many points and this could be exponential. Thus, it is an important question whether the phase reversal can be implemented efficiently. In this case, this can be achieved as the phase reversal can be implemented with symmetric functions, the implementation of which is efficient [2]. Further, as the amplitude after the Deutsch-Jozsa algorithm with (biased) Hadamard transform is known in this case, the number of iterations in Grover algorithm will be accurately estimated apriori and thus the problem of obtaining the proper stopping

criteria will not occur in this case.

VI. CONCLUSION AND OPEN PROBLEMS

In this work, we study several quantum algorithms to construct arbitrary Dicke state in a disciplined manner. The key idea is to find a suitable symmetric Boolean function for Deutsch-Jozsa algorithm for the given n and w , use of the Grover algorithm and the generalized parity measurement strategy. Further, we show that it is possible to obtain improved results using biased Hadamard transform suitably. Our results improve the probabilities obtained in [1] and thus provide faster method to construct Dicke states. The problem open in this area is to characterize the enumeration results in case of modifying the Deutsch-Jozsa algorithm with biased Hadamard transform. Obtaining the exact bias $(\frac{r}{n})$ in biased Hadamard transform with the corresponding symmetric function to optimize the probability corresponding to the Dicke state seems to be an interesting problem.

Though we look at the problem from theoretical angle, the algorithmic blocks used by us have experienced major advancement towards actual implementation. One may refer to [15, Section 7] for literature related to implementation of quantum gates as well as Deutsch-Jozsa algorithm, Grover algorithm and several measurement techniques. As example, the idea of implementing biased Hadamard transform is related to the Fabry-Perot cavity [15, Page 299].

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